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Intrinsic Lie group and nuclear collective rotation about intrinsic axes

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Abstract. For any Lie group G, a so called intrinsic Lie group $\bar{G}$ is introduced, G and $\bar{G}$ being commutative and anti-isomorphic. It is shown that in the parameter space of a Lie group G, the corresponding intrinsic Lie group $\bar{G}$ is just the second parameter group. The general relations between the first and second parameter groups are derived in a simple way.

For the group SO(3), there is the intrinsic Lie group $\overline{\mathrm{SO}}(3)$. The infinitesimal generators of $\overline{\mathrm{SO}}(3)$ are precisely the components of the angular momentum in the intrinsic coordinate system. Therefore the intrinsic Lie group $\overline{\mathrm{SO}}(3)$ provides an appropriate mathematical formalism for the description of collective rotations of nuclei about their intrinsic axes.

1. Introduction

Since the pioneering works of Rainwater (1950) and Bohr (1952) on nuclear collective rotations and vibrations, the study of these phenomena, both macroscopically and microscopically, became a subject of extreme interest (Bohr and Mottelson 1953, Elliott 1958a, b, Eisenberg and Greiner 1970, Arima and Iachello 1976, 1978a, b, and Moshinsky 1980). Recently, extensive efforts have been devoted to separating the collective degrees of freedom from the intrinsic degrees of freedom in the Hamiltonian of an A-body system with the aim to derive a collective Hamiltonian microscopically, with notable success (Vanagas 1977, 1980, Gulshani and Rowe 1976, Weaver *et al* 1976, and Gulshani 1981).

However, in this paper, rather than investigating the dynamic theory of nuclear rotations, we restrict ourselves to some kinematic aspects of nuclear collective rotations about the external as well as the intrinsic axes. As pointed out by Bohr and Mottelson (1969), the transformation of operators between the external (laboratory) and the intrinsic (moving or body-fixed) coordinate systems involves special features as a result of the fact that the orientation angles (α, β, γ) of the intrinsic frame are to be regarded as dynamical variables. One of these special features is the well known fact that the intrinsic components $J_{1,2,3}$ of total angular momentum commute with the external components $J_{x,y,z}$ and the commutation relations of the intrinsic components among themselves are similar to those of $J_{x,y,z}$, but involve an opposite sign.

Moreover, in Elliott's SU(3) rotation model (Elliott 1958a, b), or the recent interacting boson model with $\mathrm{SU}(6)\supset\mathrm{SU}(3)$ limit (Arima and Iachello 1978a), a quantum number K, the third component of total angular momentum in the intrinsic frame, is introduced to label the different rotation bands. Elliott pointed out that the intrinsic quantum number K is a rather unusual one in that the states of the same

$(\lambda\mu)L$ but different K are not orthogonal. Thus the rotation of nuclei about the intrinsic axes does involve some puzzling features.

Louck and Galbraith (1976) discussed the transformation between the laboratory system and intrinsic system. They demonstrated that the group of the classical spherical rigid rotator is a group of transformations, designated as $O_3 * O_3$ (O_3 is the three-dimensional rotation-inversion group), of two three-dimensional spaces. These two groups are anti-isomorphic, and their generators commute but have the same J^2. Similar results have also been obtained by Judd (1975) in the context of diatomic molecule rotations. This type of group structure was extended into other groups by Louck and Biedenharn (1970).

By introducing three kinds of complete set of commuting operators (CSCO), we set up a new approach to the representation theory of finite groups (Chen *et al* 1977a, b, c, 1978, 1979, and Chen and Gao 1982). The first kind of CSCO (CSCO-I) of a finite group is the analogy of the Casimir operators in Lie groups. It turns out that the CSCO-I consists of few class operators of the group and is a CSCO in its class space. Suppose $G \supset G(1) \supset G(2) \supset \ldots$ is a canonical subgroup chain of G, and C and $C(n)$ are the CSCO-I of G and $G(n)$ respectively. The set of operators

$$(C; C(s)) = (C; C(1), C(2), \ldots) \tag{1a}$$

is called the CSCO-II of G, which is a CSCO in any irreducible space of G.

For any non-abelian group G, a so called intrinsic group $\bar{G}$ is introduced (Chen *et al* 1977a, Chen and Gao 1982). $\bar{G}$ and G are commutative and anti-isomorphic (or isomorphic as well, since if $\bar{G}$ and G are anti-isomorphic, then $\bar{G}$ and G^{-1} are isomorphic, where G^{-1} is the same group as G but with all the elements R being renamed as R^{-1}). Corresponding to the group chain $G \supset G(1) \supset G(2) \supset \ldots$, we have the intrinsic group chain $\bar{G} \supset \bar{G}(1) \supset \bar{G}(2) \supset \ldots$ and the CSCO-II of $\bar{G}$,

$$(\bar{C}; \bar{C}(s)) = (\bar{C}; \bar{C}(1), \bar{C}(2), \ldots). \tag{1b}$$

It is proved that the CSCO-I of G and $\bar{G}$ are equal but not those of $G(n)$ and $\bar{G}(n)$, i.e.

$$C = \bar{C}, \qquad C(n) \neq \bar{C}(n), \qquad n = 1, 2, \ldots. \tag{1c}$$

The set of operators

$$(C, C(s), \bar{C}(s)) = (C, C(1), C(2), \ldots \bar{C}(1), \bar{C}(2), \ldots), \tag{1d}$$

is a CSCO in the group space and is termed the CSCO-III of G. It is shown that the primitive characters, the $G \supset G(1) \supset G(2) \supset \ldots$ irreducible basis and irreducible matrix elements can be easily found by solving the eigenvectors of the CSCO-I, II and III respectively. This technique, the so called eigenfunction method, proves to be very powerful in the actual calculation of characters, the Clebsch–Gordan coefficients, isoscalar factors etc (Chen and Gao 1981, 1982, Chen *et al* 1983a, b).

The close relationship between the new approach to finite groups and Racah's approach to Lie groups (Racah 1951) is apparent as far as the CSCO-I and CSCO-II are concerned. What is not yet clear is about $\bar{G}$ and $\bar{C}(s)$. We need to find out the counterparts of the intrinsic group $\bar{G}$ and the operator set $\bar{C}(s)$. We first extend the concept of the intrinsic group to the Lie group case and identify the intrinsic Lie group with the second parameter group (Racah 1951 and Eisenhart 1933). In this way we have a unified representation theory for both finite and Lie groups based on the CSCO approach. After doing this, we applied the theory of the intrinsic group to the special case of the rotation group SO(3) and showed that the intrinsic Lie group

$\overline{\mathrm{SO}}(3)$ provides an appropriate mathematical frame for describing the rotation of nuclei about the intrinsic axes (in passing, this is the reason why $\bar{G}$ is termed the intrinsic group). The conclusions reached by Louck and Galbraith (1976), Louck and Biedenharn (1970) and Bohr and Mottelson (1969) etc emerge naturally in this physically oriented approach.

2. Intrinsic Lie group

We use $R(a) \equiv R(a^1, a^2 \ldots a^r)$ to denote an element of a Lie group G of rank r, $a^1, a^2 \ldots a^r$ being r parameters. For any element $R(b)$ of the group G, similar to the case of finite groups (Chen *et al* 1977a), one can define a corresponding operator $\bar{R}(b)$ in the group space L_g by the following equation

$$\bar{R}(b)R(a) = R(a)R(b) \qquad \text{for any } R(a) \in L_g. \tag{2a}$$

It is fairly easy to prove that the set of operators $\bar{R}(b)$ forms a group, the intrinsic Lie group $\bar{G}$, which has the following properties†.

(i) The group $\bar{G}$ commutes with G,

$$[R(a), \bar{R}(b)] = 0. \tag{2b}$$

(ii) The groups $\bar{G}$ and G are anti-isomorphic, i.e. if

$$R(a)R(b) = R(c), \tag{3a}$$

then

$$\bar{R}(b)\bar{R}(a) = \bar{R}(c). \tag{3b}$$

It is seen from the definition equation $(2a)$ that for any Abelian group G the intrinsic group $\bar{G}$ coincides with the group G. In the parameter space, the infinitesimal elements of the groups G and $\bar{G}$ can be written as

$$R(\delta a) = 1 + \delta a^\rho A_\rho, \qquad \bar{R}(\delta a) = 1 + \delta a^\rho B_\rho, \tag{4}$$

where A_ρ and B_ρ represent infinitesimal operators of G and $\bar{G}$ in the parameter space, respectively. From the definition of the Lie group, we have

$$R(\delta a)R(a) = R(a + \mathrm{d}a), \qquad a^\sigma + \mathrm{d}a^\sigma = \varphi^\sigma(a, \delta a), \qquad \sigma = 1, 2, \ldots, r. \tag{5}$$

From (4) and (5) one gets

$$\begin{aligned} &A_\rho R(a) = \mu_\rho^\sigma(a)(\partial/\partial a^\sigma)R(a), \\ &A_\rho = \mu_\rho^\sigma(a)(\partial/\partial a^\sigma), \qquad \mu_\rho^\sigma(a) = \partial\varphi^\sigma(a, b)/\partial b^\rho|_{b=0}. \end{aligned} \tag{6}$$

Similarly, according to definition $(2a)$, one obtains

$$\bar{R}(\delta a)R(a) = R(a)R(\delta a) = R(a + \mathrm{d}a), \tag{7}$$

$$a^\sigma + \mathrm{d}a^\sigma = \varphi^\sigma(\delta a, a), \tag{8}$$

$$\begin{aligned} &B_\rho R(a) = \bar{\mu}_\rho^\sigma(a)(\partial/\partial a^\sigma)R(a), \\ &B_\rho = \bar{\mu}_\rho^\sigma(a)(\partial/\partial a^\sigma), \qquad \bar{\mu}_\rho^\sigma(a) = (\partial\varphi^\sigma(b, a)/\partial b^\rho)|_{b=0}. \end{aligned} \tag{9}$$

† Boerner (1963) called $\bar{G}$ the inverted regular regular representation.

Comparing (6) and (9) with Racah's equations (15) and (16) (Racah 1951) we recognise that in the parameter space the infinitesimal operators of the groups G and $\bar{G}$ are just the infinitesimal operators of the first and second parameter groups, respectively.

In physical applications one usually works with the representation of the groups G and $\bar{G}$ in the space of 'functions on the group manifold', in which the operators of the group elements are defined, by analogy with the case of finite groups, by

$$R_b u(R_a) = u(R_b^{-1} R_a), \tag{10a}$$

$$\bar{R}_b u(R_a) = u(R_a R_b^{-1}), \tag{11}$$

where $u(R_a) \equiv u(a)$ is a function on the group manifold. It is to be noted that

$$\begin{aligned} R_c R_b u(R_a) &= u[(R_c R_b)^{-1} R_a] = u(R_b^{-1} R_c^{-1} R_a) \\ &\neq R_c u(R_b^{-1} R_a) = u(R_c^{-1} R_b^{-1} R_a). \end{aligned} \tag{10b}$$

Let R_b and $\bar{R}_b$ in (10a) and (11) be infinitesimal elements,

$$\begin{aligned} &R_b = 1 + \delta b^\rho X_\rho, \qquad \bar{R}_b = 1 + \delta b^\rho \bar{X}_\rho, \\ &R_b^{-1} = 1 - \delta b^\rho X_\rho, \end{aligned} \tag{12}$$

where we use X_ρ and $\bar{X}_\rho$ to denote the infinitesimal operators of the groups G and $\bar{G}$, respectively. With the help of (10)–(12), using the same procedure which led to (6) and (9), one gets expressions for the operators X_ρ and $\bar{X}_\rho$ when they act on the function on the group manifold.

$$X_\rho(a) = -\mu_\rho^\sigma(a)(\partial/\partial a^\sigma) = -A_\rho(a), \qquad \bar{X}_\rho(a) = -\bar{\mu}_\rho^\sigma(a)(\partial/\partial a^\sigma) = -B_\rho(a). \tag{13}$$

It is seen that the infinitesimal operators $X_\rho(a)$ and $\bar{X}_\rho(a)$ in the space of the function on the group manifold are identical to the infinitesimal operators A_ρ and B_ρ in the parameter space, respectively, except for a trivial difference in sign which results from the fact that in (10a) and (11) we used the definition $R_b u(R_a) = u(R_b^{-1} R_a)$ instead of $R_b u(R_a) = u(R_b R_a)$, etc.

In future we mainly concern ourselves with $X_\rho(a)$ and $\bar{X}_\rho(a)$, and for convenience we shall just call them operators of the first and second parameter groups.

The irreducible matrix elements $D_{mk}^{(\nu)}(R_a) \equiv D_{mk}^{(\nu)}(a)$ of the group may be thought of as a function on the group manifold. Substituting $D_{mk}^{(\nu)}(a)$ for $u(a)$ in (10) and (11), and using (12) and (13), we have

$$X_\rho D_{mk}^{(\nu)}(a) = -\sum_{m} (m|X_\rho|m')^{(\nu)} D_{m'k}^{(\nu)}(a), \tag{14a}$$

$$\bar{X}_\rho D_{mk}^{(\nu)}(a) = -\sum_{k'} (k'|X_\rho|k)^{(\nu)} D_{mk'}^{(\nu)}(a), \tag{14b}$$

with

$$(m|X_\rho|t)^{(\nu)} = D_{mt}^{(\nu)}(X_\rho). \tag{15}$$

In Chen *et al* (1977c) it was proved that

$$D_{mt}^{(\nu)}(R_a) = D_{tm}^{(\nu)}(\bar{R}_a). \tag{16}$$

From (12) and (16) one gets

$$(m|X_\rho|t)^{(\nu)} = (t|\bar{X}_\rho|m)^{(\nu)}. \tag{17}$$

Alternatively, by letting $u(R_a)$ in (10a) and (11) be

$$u(R_a)=\tilde{D}^{-1(\nu)}_{mk}(R_a)=D^{(\nu)*}_{mk}(a),$$

one obtains

$$X_\rho D^{(\nu)*}_{mk}(a)=\sum_{m'}(m'|X_\rho|m)^{(\nu)}D^{(\nu)*}_{m'k}(a), \tag{18a}$$

$$\bar{X}_\rho D^{(\nu)*}_{mk}(a)=\sum_{k'}(k|X_\rho|k')^{(\nu)}D^{(\nu)*}_{mk'}(a). \tag{18b}$$

3. Relations between first and second parameter groups

Because of properties (2b) and (3), one gets for the infinitesimal operators

$$[X_\tau,\bar{X}_\rho]=0, \tag{19}$$

$$[X_\tau,X_\rho]=C^\sigma_{\tau\rho}X_\sigma,\qquad [\bar{X}_\tau,\bar{X}_\rho]=-C^\sigma_{\tau\rho}\bar{X}_\sigma,\qquad \tau,\sigma=1,2,\ldots,r. \tag{20}$$

According to definition (2a), the relation between the element $\bar{R}(b)$ of the intrinsic group and the element $R(b)$ of the group G is seen to be

$$\bar{R}(b)=R(a)R(b)R^{-1}(a). \tag{21a}$$

It should be emphasised that equation (2a) is the defining equation for the operator $\bar{R}(b)$, rather than an identity equation. Therefore, it is not permissible to multiply equation (2a) from the right by another group element $R(c)$, i.e.

$$\bar{R}(b)R(a)R(c)\neq R(a)R(b)R(c).$$

Instead, $R(a)R(c)$ must be considered as a new element of G,

$$\bar{R}(b)R(a)R(c)=\bar{R}(b)(R(a)R(c))=R(a)R(c)R(b).$$

Analogously, equation (21a) is also not an identity; it only shows that $\bar{R}(b)$ is equivalent to $R(a)R(b)R^{-1}(a)$ when acting on $R(a)$, while acting on another element $R(c)$, it will be equivalent to $R(c)R(b)R^{-1}(c)$. In other words, the element $R(a)$ in equation (21a) is a variable one rather than a fixed one; it changes according to the 'basis' on which $\bar{R}(b)$ acts.

By using equations (11) and (10b), it is easy to show that equation (21a) holds when acting on any function $u(a)$ on the group manifold

$$\bar{R}(b)u(a)=R(a)R(b)R^{-1}(a)u(a). \tag{21b}$$

Equation (21b) is an identity in the sense that it holds for any parameters a and b. Letting the parameter b be infinitesimal, inserting (12) into (21b), and remembering that acting on the function $u(a)$, $X_\rho\to X_\rho(a)$, and $\bar{X}_\rho\to\bar{X}_\rho(a)$, we get

$$\bar{X}_\rho(a)u(a)=R(a)X_\rho(a)R^{-1}(a)u(a). \tag{21c}$$

Since $u(a)$ is an arbitrary function on the group manifold, it follows that

$$\bar{X}_\rho(a)=R(a)X_\rho(a)R^{-1}(a). \tag{22a}$$

Equation (22a) gives the relations between the infinitesimal operators of the first and second parameter groups. It is worth noting that (22a) is an identity relation.

To get a geometric interpretation of (22a), let the r infinitesimal operators X_ρ be thought of as the r components of an abstract vector X in a fixed coordinate system of an r-dimensional vector space, and X'_ρ be the new components of the vector X in another fixed system rotated through a given 'angle' a_0 with respect to the original one. Obviously we have

$$X'_\rho = R(a_0)XR^{-1}(a_0). \tag{23a}$$

We know that the X_ρ carry the adjoint representation $\mathscr{D}^{(\nu_0)}$ of the Lie group G, thus

$$X'_\rho = \sum_\sigma \mathscr{D}^{(\nu_0)}_{\sigma\rho}(a_0)X_\sigma. \tag{23b}$$

By comparing (22a) with (23a), we obtain

$$\bar{X}_\rho(a) = \sum_\sigma \mathscr{D}^{(\nu_0)}_{\sigma\rho}(a)X_\sigma(a). \tag{22b}$$

The inverse of (22b) is

$$X_\sigma(a) = \sum_\rho \mathscr{D}^{(\nu_0)*}_{\sigma\rho}(a)\bar{X}_\rho(a). \tag{22c}$$

It must be stressed that although (22) and (23) are similar in appearance, actually they are drastically different. In (23) the parameter a_0 is a constant and (23b) shows that X'_ρ and X_ρ are members of the same Lie algebra. Another way of saying this is that the Lie algebras $\{X'_\rho\}$ and $\{X_\rho\}$ are isomorphic but not commutative. On the contrary, we know that $\bar{X}_\rho(a)$ and $X_\rho(a)$ belong to the Lie algebras of the first and second parameter groups respectively, which are anti-isomorphic but commutative. The reason for the 'strange' behaviour of $\bar{X}_\rho(a)$ is that the parameter a in (22) is a dynamic variable instead of a constant. Although $\bar{X}_\rho(a)$ is a linear combination of $X_\rho(a)$ as shown in (22b), $\bar{X}_\rho(a)$ and $X_\rho(a)$ are not members of the same Lie algebra, since the linear combination coefficients $\mathscr{D}^{(\nu_0)}_{\sigma\rho}(a)$ are a-dependent functions.

As mentioned in the introduction, the intrinsic components $J_{1,2,3}$ of angular momentum J have similar behaviour. This itself suggests that we might regard the differential operators $\bar{X}_\rho(a)$ as the explicit form of the components $\bar{X}_\rho$ of the vector X in an intrinsic coordinate system of the r-dimensional vector space, the orientation 'angle' a of which is a dynamical variable. Consequently, without specifying the action space, (22a) can be rewritten as

$$\bar{X}_\rho = R(a)X_\rho R^{-1}(a), \tag{22d}$$

which gives the general relation between the infinitesimal operators of the Lie group G and its intrinsic group $\bar{G}$. Hence we see that (23) represents a transformation from a fixed frame to another fixed frame, while (22) represents a transformation from a fixed frame to a moving (or intrinsic) frame.

Finally, we want to point out that the relation (22b) between the infinitesimal operators of the first and second parameter groups is a generalisation of Eisenhart's equation (14.10) (Eisenhart 1933). Letting the group element $R(a_0)$ in equation (23) be an infinitesimal one

$$R(a_0) = 1 + \delta a_0^\tau X_\tau, \tag{24}$$

one gets

$$[X_\tau, X_\rho] = \sum_\sigma \mathscr{D}^{(\nu_0)}_{\sigma\rho}(X_\tau)X_\sigma. \tag{25}$$

Accordingly

$$C^{\sigma}_{\tau\rho} = \mathscr{D}^{(\nu_0)}_{\sigma\rho}(X_\tau). \tag{26}$$

Assuming a to be regular parameters (Eisenhart 1933), for small a,

$$R(a) = \exp(-a^\tau X_\tau) = 1 - a^\tau X_\tau + \frac{1}{2!}a^\tau a^{\tau'} X_\tau X_{\tau'} - \ldots. \tag{27}$$

From (26) and (27) we have

$$\mathscr{D}^{(\nu_0)}_{\sigma\rho}(a) = \delta^\sigma_\rho - a^\tau C^\sigma_{\tau\rho} + \frac{1}{2!}a^\tau a^{\tau'} C^\sigma_{\tau\mu} C^\mu_{\tau'\rho} - \ldots. \tag{28}$$

Equations ($22b$) and (28) are precisely Eisenhart's equations (14.10) and (14.13).

4. The CSCO-I, II and III of a Lie group

For a Lie group G of rank l and order r, the CSCO-I is defined as

$$C = (I_l(X_\rho), \ldots I_1(X_\rho)), \tag{29}$$

where $I_i(X_\rho)$ are the Casimir invariants of G. The CSCO-I of the intrinsic Lie group $\bar{G}$ is

$$\bar{C} = (I_l(\bar{X}_\rho), \ldots I_1(\bar{X}_\rho)). \tag{30}$$

The conclusion obtained for finite groups that the CSCO-I of G and $\bar{G}$ are equal also applies to the Lie group. To show this one only needs to prove that a Lie group G and its intrinsic group $\bar{G}$ have the same Casimir invariants; i.e.

$$I_i(X_\rho) = I_i(\bar{X}_\rho), \qquad i = 1, 2, \ldots l. \tag{31}$$

Proof. According to the definition of the Casimir invariants, $I_i(X_\rho)$ commute with any element of G

$$R(a) I_i(X_\rho) R^{-1}(a) = I_i(X_\rho). \tag{32}$$

Besides, we know that $I_i(X_\rho)$ are polynomials of the infinitesimal operators X_ρ. Using equation ($22d$) we thus have

$$R(a) I_i(X_\rho) R^{-1}(a) = I_i(\bar{X}_\rho). \tag{33}$$

Therefore (31) holds.

Suppose $G \supset G(1) \supset G(2) \supset \ldots$ is a canonical subgroup chain, and

$$C(n) = (I^{(n)}_{l_n}(X_\rho), \ldots I^{(n)}_1(X_\rho)) \tag{34}$$

is the CSCO-I of the subgroup $G(n)$ with rank l_n, then the CSCO-I of the subgroup $\bar{G}(n)$ of the intrinsic group $\bar{G}$ is given by

$$\bar{C}(n) = (I^{(n)}_{l_n}(\bar{X}_\rho), \ldots I^{(n)}_1(\bar{X}_\rho)). \tag{35}$$

Obviously $\bar{C}(n)$ commutes with but is not equal to $C(n)$, since $\bar{C}(n)$ commutes with the whole group G, while $C(n)$ commutes only with the subgroup $G(n)$.

The CSCO-II and III are given by (1a) and (1d). Theorems 1–6 given by Chen *et al* (1977a) for finite groups are still valid for compact Lie groups. Similarly, the eigenvalues of the CSCO-II can be used to label irreducible basis vectors, while the eigenvalues of the intrinsic operator set $\bar{C}(s) = (\bar{C}(1), \bar{C}(2), \ldots)$ can be used to distinguish between repeated irreducible representations of G.

It is thus seen that for the representations of both finite groups and compact Lie groups we can have a unified treatment based on the commuting operator approach of quantum mechanics, which is more acceptable to physicists.

5. Irreducible tensor of the intrinsic Lie group

In this section, the ordinary tensor algebra of a Lie group G will be extended to the intrinsic Lie group $\bar{G}$. The ordinary irreducible tensor is defined as

$$[X_\rho, T_m^{(\nu)}] = \sum_m D_{m'm}^{(\nu)}(X_\rho) T_{m'}^{(\nu)} \tag{36}$$

where $D^{(\nu)}(X_\rho)$ stands for the irreducible matrix. It is to be noted that here the irreducible matrix $D^{(\nu_0)}$ of the adjoint representation is not necessarily identical with $\mathscr{D}^{(\nu_0)}$ of (25). For a given $D^{(\nu_0)}$, only through a proper linear combination of X_ρ can we combine them into the ν_0-irreducible tensor $T_m^{(\nu_0)}$. Henceforth it is assumed that the infinitesimal operators X_ρ have been chosen to be identical with $T_\rho^{(\nu_0)}$. With this provision, all the equations (22)–(28) remain valid when the matrix $\mathscr{D}^{(\nu_0)}$ is replaced by $D^{(\nu_0)}$.

By analogy with the fact that X_ρ is the ν_0-irreducible tensor of the group G, it is natural to define $\bar{X}_\rho$ as the ν_0-irreducible tensor of the intrinsic Lie group $\bar{G}$. From (20) and (25) we have

$$[\bar{X}_\tau, \bar{X}_\rho] = -\sum_\sigma D_{\sigma\rho}^{(\nu_0)}(X_\tau) \bar{X}_\sigma. \tag{37}$$

Therefore a general definition of the ν-irreducible tensor $\bar{T}_k^{(\nu)}$ of the intrinsic group $\bar{G}$ is

$$[\bar{X}_\tau, \bar{T}_k^{(\nu)}] = -\sum_{k'} D_{k'k}^{(\nu)}(X_\tau) \bar{T}_{k'}^{(\nu)}. \tag{38a}$$

Comparing (18a) with (36), and (14b) with (38a), we know that $D_{mk}^{(\nu)*}(a)$ is the mth component of the νth irreducible tensor for the group G, while $D_{mk}^{(\nu)}(a)$ is the kth component of the νth irreducible tensor for the intrinsic group $\bar{G}$:

$$D_{mk}^{(\nu)*}(a) = T_m^{(\nu)}, \qquad D_{mk}^{(\nu)}(a) = \bar{T}_k^{(\nu)}. \tag{38b}$$

From (38b) it is seen that there are h_ν independent tensor operators $T^{(\nu)1}, T^{(\nu)2}, \ldots T^{(\nu)h_\nu}$ with the same irrep label (ν), which are enumerated by assigning the intrinsic quantum k to each of them. This is consistent with a theorem proved by Louck and Biedenharn (1970). The theorem asserts that the number of the linearly independent irreducible tensor operators with the irrep label (ν) is equal to the dimension h_ν of the irrep (ν) of G. In their notation

$$D_{mk}^{(\nu)*}(a) = \left\langle \begin{matrix} k \\ (\nu) \\ m \end{matrix} \right\rangle.$$

Similarly, we have h_ν linearly independent irreducible tensors of the intrinsic group $\bar{G}$ with the irrep label (ν),

$$D^{(\nu)}_{mk}(a) = \left\langle \begin{matrix} m \\ (\nu) \\ k \end{matrix} \right\rangle.$$

It is thus seen that the same conclusions are reached from the viewpoint of the intrinsic group.

Applying (36)–(38), (14), (17) and (18) to the special case of the SO(3) group, we can easily reproduce all the results given by Bohr and Mottelson (1969, Section 1A-6).

6. Intrinsic group and intrinsic state of SO(3)

We shall use the SO(3) group as an example to illustrate the physical meaning of the first and second parameter groups as well as the intrinsic group.

6.1. *The intrinsic group $\overline{SO}(3)$*

By choosing the Euler angles α, β, γ as group parameters, the elements of SO(3) can be written

$$R(\alpha, \beta, \gamma) = \exp(-i\alpha J_z) \exp(-i\beta J_y) \exp(-i\gamma J_z). \tag{39a}$$

On account of the anti-isomorphic property of the intrinsic group $\overline{SO}(3)$ with respect to SO(3), the elements of $\overline{SO}(3)$ are of the form

$$\bar{R}(\alpha, \beta, \gamma) = \exp(-i\gamma \bar{J}_z) \exp(-i\beta \bar{J}_y) \exp(-i\alpha \bar{J}_z). \tag{39b}$$

The infinitesimal operators J_x, J_y, J_z form the basis of the adjoint ($J = 1$) representation of the group SO(3) in which the representative of the group element $R(\alpha, \beta, \gamma)$ is $\mathscr{D}^{(\nu_0)}(\alpha, \beta, \gamma)$; the transposition of $\mathscr{D}^{(\nu_0)}(\alpha, \beta, \gamma)$ is

$$\tilde{\mathscr{D}}^{(\nu_0)}(\alpha, \beta, \gamma) = M(\alpha, \beta, \gamma)$$

$$= \begin{pmatrix} \cos\alpha \cos\beta \cos\gamma - \sin\alpha \sin\gamma, & \sin\alpha \cos\beta \cos\gamma + \cos\alpha \sin\gamma, & -\sin\beta \cos\gamma \\ -\cos\alpha \cos\beta \sin\gamma - \sin\alpha \cos\gamma, & -\sin\alpha \cos\beta \sin\gamma + \cos\alpha \cos\gamma, & \sin\beta \sin\gamma \\ \cos\alpha \sin\beta & \sin\alpha \sin\beta & \cos\beta \end{pmatrix}. \tag{40a}$$

The relation between the infinitesimal operators $\bar{J}_{x,y,z}$ of the intrinsic group $\overline{SO}(3)$ and $J_{x,y,z}$ of the group SO(3) follows from equation (22*b*):

$$\begin{pmatrix} \bar{J}_x \\ \bar{J}_y \\ \bar{J}_z \end{pmatrix} = M(\alpha, \beta, \gamma) \begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix}. \tag{40b}$$

We now proceed to calculate the infinitesimal operators of the first parameter group by employing equation (13). Consider the product of two successive rotations

$$R(\alpha_3, \beta_3, \gamma_3) = R(\alpha_2, \beta_2, \gamma_2) R(\alpha_1, \beta_1, \gamma_1).$$

As the combination laws for the parameters α, β, γ are not easy to obtain explicitly, we first give the combination laws for the parameters of the group SU(2)

$$\begin{pmatrix} c_0 - \mathrm{i}c_3, & -c_1 - \mathrm{i}c_2 \\ c_1 - \mathrm{i}c_2, & c_0 + \mathrm{i}c_3 \end{pmatrix} = \begin{pmatrix} b_0 - ib_3, & -b_1 - ib_2 \\ b_1 - ib_2, & b_0 + ib_3 \end{pmatrix} \begin{pmatrix} a_0 - \mathrm{i}a_3, & -a_1 - \mathrm{i}a_2 \\ a_1 - \mathrm{i}a_2, & a_0 + \mathrm{i}a_3 \end{pmatrix}. \tag{41}$$

In terms of the relations between the parameters α, β, γ of SO(3) and the parameters a_0, a_1, a_2, a_3 of SU(2) (Smirnov 1951)

$$\begin{aligned} a_0 &= \cos \tfrac{1}{2}\beta_1 \cos \tfrac{1}{2}(\alpha_1 + \gamma_1), & a_1 &= \sin \tfrac{1}{2}\beta_1 \cos \tfrac{1}{2}(\gamma_1 - \alpha_1), \\ a_2 &= \sin \tfrac{1}{2}\beta_1 \sin \tfrac{1}{2}(\gamma_1 - \alpha_1), & a_3 &= \cos \tfrac{1}{2}\beta_1 \sin \tfrac{1}{2}(\alpha_1 + \gamma_1), \end{aligned} \tag{42}$$

one then gets implicitly the combination laws for the parameters α, β, γ. From equations (13), (41) and (42) we obtain

$$J_z = \mathrm{i}^{-1}(\partial/\partial\alpha), \tag{43a}$$

$$J_y = \mathrm{i}^{-1}[-\sin\alpha \cot\beta(\partial/\partial\alpha) + \cos\alpha(\partial/\partial\beta) + (\sin\alpha/\sin\beta)(\partial/\partial\gamma)]. \tag{43b}$$

So far we have only got the differential operators J_y and J_z, since X_α and X_γ correspond to the same J_z. The third operator J_x can be obtained by using the commutator

$$\begin{aligned} &[J_y, J_z] = \mathrm{i}J_x. \\ &J_x = \mathrm{i}^{-1}[-\cos\alpha \cot\beta(\partial/\partial\alpha) - \sin\alpha(\partial/\partial\beta) + (\cos\alpha/\sin\beta)(\partial/\partial\gamma)]. \end{aligned} \tag{43c}$$

Comparing equation (43) with Eisenberg's equation (39) (see Eisenberg and Greiner 1970) we know that, for SO(3), the infinitesimal operators of the first parameter group are nothing else but the well known differential operators J_x, J_y, J_z, the components of the angular momentum in the fixed frame, when acting on functions on the group manifold.

Similarly, from equations (13), (41) and (42) one gets the explicit form of the infinitesimal operators of the intrinsic group $\overline{\mathrm{SO}}(3)$, i.e. the infinitesimal operators of the second parameter group

$$\begin{aligned} \bar{J}_x &= \mathrm{i}^{-1}[\cos\gamma \cot\beta(\partial/\partial\gamma) - \sin\gamma(\partial/\partial\beta) - (\cos\gamma/\sin\beta)(\partial/\partial\alpha)] = -J_x(\alpha \leftrightarrow \gamma), \\ \bar{J}_y &= \mathrm{i}^{-1}[-\sin\gamma \cot\beta(\partial/\partial\gamma) + \cos\gamma(\partial/\partial\beta) + (\sin\gamma/\sin\beta)(\partial/\partial\alpha)] = J_y(\alpha \leftrightarrow \gamma), \\ \bar{J}_z &= \mathrm{i}^{-1}(\partial/\partial\gamma) = J_z(\alpha \leftrightarrow \gamma). \end{aligned} \tag{44}$$

It can be easily verified that equations (43) and (44) are consistent with equation (40).

Comparing equation (44) with Eisenberg's equation (29) it is seen that the infinitesimal operators $\bar{J}_{x,y,z}$ of the intrinsic group $\overline{\mathrm{SO}}(3)$ are just the components of the angular momentum $J_{1,2,3}$ in the intrinsic frame. We thus conclude that *the components of angular momentum in the intrinsic frame* J_1, J_2, J_3 *constitute the generators of the intrinsic group* $\overline{\mathrm{SO}}(3)$, just as the components of angular momentum in the fixed frame, J_x, J_y, J_z constitute the generators of SO(3).

6.2. Intrinsic state

Equation (2*a*) only defines the action of the group element $\bar{R}(b)$ in the group space of G. The action of the intrinsic group element $\bar{R}(a)$ in the configuration space is yet to be specified. It leads to the definition of the so called 'intrinsic state'. If there is a set of wavefunctions $\{\Phi_a(X)\}$ in the configuration space which span a reducible representation of G, then one may pick up any one of them, $\Phi_0(X)$ say, and define the action of any intrinsic group element $\bar{R}(a)$ on this chosen state $\Phi_0(X)$ to be identical to the action of the group element $R(a)$ of G

$$\bar{R}(a)\Phi_0(X) = R(a)\Phi_0(X), \qquad \text{for all } R(a) \in G \tag{45}$$

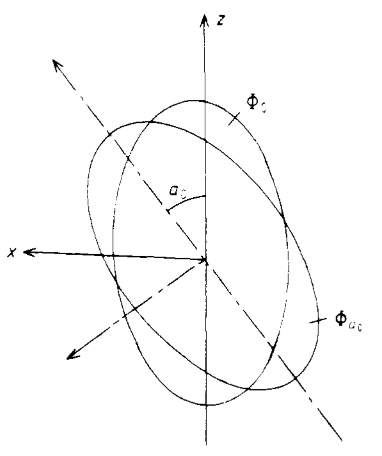
and $\Phi_0(X)$ is called the intrinsic state of the group G.

Equations ($2a$) and (45) suffice to define the action of the intrinsic group elements on all other wavefunctions $\Phi_a(X)$ which can be obtained from $\Phi_0(X)$ by acting with group element $R(a)$ of G

$$\begin{aligned} \Phi_a(X) &= R(a)\Phi_0(X), \\ \bar{R}(b)\Phi_a(X) &= R(a)R(b)\Phi_0(X). \end{aligned} \tag{46}$$

For the group $\overline{\mathrm{SO}}(3)$ (SO(3)), $\bar{R}(a)(R(a))$ is a rotation operator about the moving (fixed) axes of the intrinsic (external) frame, a being the Euler angles (α, β, γ). *In the case when the intrinsic axes coincide with the external ones, the wavefunction in the fixed frame, namely the wavefunction $\Phi_0(X)$ in the configuration space, is just our intrinsic state*, and the rotations $\bar{R}(a)$ about the intrinsic axes are identical to the rotations $R(a)$ about the external axes. That is what equation (45) means. To further elucidate the physical meaning of the intrinsic state, we take the collective rotation of a deformed nucleus as an example. An HF state of a nucleus with an open shell is in general nonspherical, say oblate; we choose the symmetry axes of the oblate spheroid as the intrinsic axes. *Denote the HF state whose symmetry axes coincide with the external ones (i.e. the coordinate axes of the fixed frame) by* $\Phi_0(X)$ and the others by $\Phi_a(X)$ which can be generated from $\Phi_0(X)$ by acting with a rotation operator $R(a)$ (see equation (46)), the parameters a being the orientation angles of the intrinsic frame. From the set $\{\Phi_a(X)\}$, we choose $\Phi_0(X)$ as the intrinsic state of SO(3). On account of the definition of the function $\Phi_0(X)$, (45) obviously holds.

It was pointed out (Chen 1983) that among the states $\{\Phi_a(X)\}$, in principle we are free to choose any one as the intrinsic state. For example we may choose $\Phi_{a_0}(X) = R(a_0)\Phi_0(X)$ as the new intrinsic state whose symmetry axes have been rotated by an angle a_0 from the external ones (see the schematic diagram below). According to the definition of (45), the intrinsic state is a state whose intrinsic axes coincide with the external ones, so it means that the newly chosen intrinsic axes no longer coincide with



the symmetry axes. Thus the arbitrariness in the choice of the intrinsic state is a reflection of the arbitrariness in the choice of intrinsic axes. These intrinsic states in general do not have a definite angular momentum but they do have a definite z component of the angular momentum, K say. From (45) one has

$$\bar{J}_z \Phi_0^{(K)}(X) = J_z \Phi_0^{(K)}(X) = K \Phi_0^{(K)}(X). \tag{47}$$

This implies that the z component of angular momentum of the HF state $\Phi_0(X)$ whose symmetry axes coincide with the external axes is just the third component of the angular momentum in the intrinsic frame.

6.3. CSCO-III of group SO(3)

By a straightforward generalisation of the results in Chen *et al* (1978, 1983a, b), we know that $(J^2, J_z, \bar{J}_z)$ constitute the CSCO-III of the group SO(3) whose eigenoperator $P_M^{(J)K}$ is the generalised projection operator of SO(3).

$$\begin{pmatrix} J_2 \\ J_z \\ \bar{J}_z \end{pmatrix} P_M^{(J)K} = \begin{pmatrix} J(J+1) \\ M \\ K \end{pmatrix} P_M^{(J)K}, \tag{48}$$

$$P_M^{(J)K} = \frac{2J+1}{8\pi^2} \int D_{MK}^{(J)*}(\alpha, \beta, \gamma) R(\alpha, \beta, \gamma) \sin\beta \, \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}\gamma. \tag{49}$$

The action of $P_M^{(J)K}$ on the intrinsic state $\Phi_0^{(K)}(X)$ yields a state with definite angular momentum JM, if it does not vanish. Now consider the question of the intrinsic component of the total angular momentum. It was pointed out (Chen *et al* 1983a, b) that not every intrinsic group element has a definite action on the states in the configuration space (unless they form a regular representation of the group), only the class operators of certain intrinsic subgroups have definite actions. The method of finding these class operators was given in Chen *et al* (1983a, b), namely, first find out all the operators which leave $\Phi_0^{(K)}(X)$ unchanged. For $\Phi_0^{(K)}(X)$ of (47), this operator is seen to be $R'_z(\varphi) = \exp[-\mathrm{i}\varphi(J_z - K)]$

$$R'_z(\varphi)\Phi_0^{(K)}(X) = \Phi_0^{(K)}(X).$$

This means that $\Phi_0^{(K)}(X)$ is an axial symmetry state. Therefore for any state $R(\alpha, \beta, \gamma)\Phi_0^{(K)}(X)$ only the intrinsic group element $\bar{R}_z(\varphi) = \exp(-\mathrm{i}\bar{J}_z\varphi)$, and thus the operator $\bar{J}_z$, have definite meaning. From (48), (19), (47)

$$\begin{aligned} \bar{J}_z P_M^{(J)K'} \Phi_0^{(K)}(X) &= (\bar{J}_z P_M^{(J)K'})\Phi_0^{(K)}(X) = K' P_M^{(J)K'} \Phi_0^{(K)}(X) \\ &= P_M^{(J)K'} J_z \Phi_0^{(K)}(X) = K P_M^{(J)K'} \Phi_0^{(K)}(X). \end{aligned}$$

Therefore

$$\begin{aligned} &P_M^{(J)K'} \Phi_0^{(K)}(X) = \delta_{kk'} \Psi_M^{(J)K}, \\ &\Psi_M^{(J)K} = \frac{2J+1}{8\pi^2} \int D_{MK}^{(J)*}(\alpha\beta\gamma) R(\alpha\beta\gamma) \Phi_0^{(K)}(X) \sin\beta \, \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}\gamma. \end{aligned} \tag{50}$$

Equations (50) show that for axial symmetry states, the third component of the total angular momentum in the intrinsic frame is determined entirely by the third component of the angular momentum of the intrinsic state. In other words, there are no collective rotations about the symmetry axis (Bohr and Mottelson 1969). Thus the projected states obey the simultaneous eigenequation

$$\begin{pmatrix} J^2 \\ J_z \\ \bar{J}_z \end{pmatrix} \Psi_M^{(J)K}(X) = \begin{pmatrix} J(J+1) \\ M \\ K \end{pmatrix} \Psi_M^{(J)K}(X). \tag{51}$$

In Elliott's SU(3) model, the so called leading state $\phi((\lambda\mu)\varepsilon\Lambda\nu)$ is chosen as the intrinsic state with $\varepsilon = 2\lambda + \mu$, $\Lambda = \frac{1}{2}\mu$, $\nu = \mu$; ϕ does not have a definite K value but contains a series of K values: $K = \mu, \mu - 2, \ldots 1$ or 0. Using the projection operator one can likewise pick out the JMK component $P_M^{(J)K}\phi((\lambda\mu)\varepsilon\Lambda\nu)$.

Another point worth mentioning is the fact that the projected states $\Psi_M^{(J)K}(X)$ are not orthogonal in the intrinsic quantum number K unless Φ_0 is a basis of the regular representation (for example $\Psi_M^{(J)K} = D_{MK}^J(\alpha, \beta, \gamma)$ are orthogonal in K). The non-orthogonality in the intrinsic quantum number is a general phenomenon. However, for finite groups, the difficulty of non-orthogonality in the intrinsic quantum number can be ultimately overcome (Chen and Gao 1982).

At last we return to the physical meaning of the intrinsic irreducible tensor.

From (14*b*), (52), it can be shown that if the irreducible tensor $T_m^{(\nu)}$ of the group SO(3) is a scalar under rotations about the intrinsic axes

$$[\bar{J}_\rho, T_m^{(\nu)}] = 0, \tag{52}$$

then the tensor

$$T_m'^{(\nu)} = \sum_{m'} D_{m'm}^{(\nu)}(\alpha, \beta, \gamma) T_{m'}^{(\nu)} \tag{53}$$

is an intrinsic irreducible tensor $\bar{T}_m^{(\nu)}$. $T_m'^{(\nu)}$ is nothing else but the operator $T_m^{(\nu)}$ expressed in the intrinsic frame. Some important operators do satisfy the condition (52): for example, the angular momentum and the multipole operators.

7. Summary

The first and second parameter groups have been known for a long time. However, on account of their being rather abstract, little attention has been paid to them. Nothing was mentioned about their use in physical problems. It was considered to be a mathematical trick of not much use. So it is rather surprising to find that in nuclear physics we have come across the realisation of these abstract ideas many times without being able to recognise them. In fact, for the group SO(3), the infinitesimal operators of the first (second) parameter group are just the differential operators of J_x, J_y, J_z $(\bar{J}_x, \bar{J}_y, \bar{J}_z)$, the components of angular momentum in the fixed (intrinsic) frame, acting on 'functions on the group'. The relationship between these two kinds of parameter groups emerges naturally as soon as their physical implications are made clear.

We abstract the concept of the intrinsic group from the concrete physical problem and then use it again in treating the problem of the collective rotation of nuclei about the intrinsic axes. Some puzzling aspects of the problem now become clear. So it may be said that we have tailored a mathematical formalism for describing nuclear rotation about the intrinsic axes.

As is well known, there is a gap between the representation theories of the finite group and Lie group (Gamba 1969, Killingbeck 1970, 1973). It is not unusual for a person familiar with one of them to be ignorant of the other. The new approach to finite group representation with its natural extension, presented here, to the compact Lie group, provides a simple and unified theory for both of them. Besides the advantage in practical calculation, the new approach is also valuable in methodology since one can learn the representation theory of the Lie group (rather formidable for novices) from the much easier one of the finite group.

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